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HOMEOMORPHISMS THAT INDUCE MONOMORPHISMS OF SOBOLEV SPACES

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ABSTRACT

Let G, G' be domains in $\mathbb{R}^n$. We obtain a geometrical description of the class of all homeomorphisms $\varphi : G \to G'$ that induce bounded operators φ^* from the seminormed Sobolev space $L^1_p(G')$ to $L^1_p(G)$ by the rule $\varphi^* u = u \circ \varphi$. For p-Poincare domains the same classes of homeomorphisms induce bounded operators for classical Sobolev spaces W^1_p. These classes of homeomorphisms are natural generalizations of the class of quasiconformal homeomorphisms that correspond to the case $p = n$. We demonstrate some applications of our results for embedding theorems in domains with Hölder singularities.

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1. Introduction

Let G be a domain in R^n. A function $u\colon G \to R$ belongs to the space $L^1_p(G)$ if u is weakly differentiable and $\|\nabla u\|_{L_p(G)} < \infty$. The linear space $L^1_p(G)$ is a seminormed space with the seminorm

$$\|u\|_{L^1_p(G)} \stackrel{\text{def}}{=} \|\nabla u\|_{L_p(G)}.$$

A function $u\colon G \to R$ belongs to the space $W^1_p(G)$ if $u \in L_p(G) \cap L^1_p(G)$. The linear space $W^1_p(G)$ is a Banach space with the norm

$$\|u\|_{W^1_p(G)} \stackrel{\text{def}}{=} \left\{\|u\|^2_{L_p(G)} + \|u\|^2_{L^1_p(G)}\right\}^{1/2}.$$

We will also use the short notation $\|u\|_{1,p}$ instead of $\|u\|_{L^1_p(G)}$ and $\|u\|_{(1,p)}$ instead of $\|u\|_{W^1_p(G)}$.

Let G and G' be two domains in R^n.

Problem: Change of variable problem for Sobolev spaces. Describe the set of all homeomorphisms $\varphi\colon G \to G'$ that induce bounded operators $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ $(\varphi^*\colon W^1_p(G') \to W^1_p(G))$.

We use the notation $\varphi^*(u) = u \circ \varphi$.

The problem was studied in the papers [16], [17], and [5] for invertible operators φ^*. The following result was established:

THEOREM 1.1: * *Let G and G' be two domains in $\mathbb{R}^n$ and let $\varphi\colon G \to G'$ be a measurable mapping. Suppose that the mapping φ induces a bounded operator $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ such that the operator $(\varphi^*)^{-1}$ is also bounded. Then:*

(1) *For $p = n$, the mapping $\varphi\colon G \to R^n$ is a quasiconformal homeomorphism.*

(2) *For $p > n-1, p \neq n$, the mapping $\varphi\colon G \to R^n$ is a quasiisometrical homeomorphism.***

The inverse theorem is much simpler to establish. A quasiconformal homeomorphism $\varphi\colon G \to G'$ induces a bounded invertible operator φ^* for $p = n$ (see, for example, [12]). A quasiisometrical homeomorphism φ induces a bounded operator for all p (see, for example, [4], [19]).

* We formulate a simple form of a more general result. The general form is to be found in [4] (chapter 4).

** For $p \leq n-1$ the result is true if the mapping φ is continuous. We could not prove these results for measurable mappings. The definition of quasiisometrical homeomorphisms can be found in Section 2.

Notation: As usually $B(x,r)$ is an open ball with center x and radius r, $B_r \stackrel{\text{def}}{=} B(0,r)$, $S(x,r)$ is a sphere with center x and radius r. If $A \subset R^n$ is an arbitrary set, then ∂A is the boundary of A, $\mid A \mid$ is the outer Lebesgue measure of A, $\operatorname{cl} A$ is the closure of A and $\operatorname{Int} A$ is the interior of A.

Let

$$\varphi_v(x,r) = \frac{\mid \varphi(B(x,r)) \mid}{\mid B(x,r) \mid}.$$

We use the following quantities:

The volume derivative $\varphi_v'(x)$ of the homeomorphism φ at the point x

$$\varphi_v'(x) = \lim_{r\to 0} \varphi_v(x,r),$$

the local Lipschitz constant of a homeomorphism φ in the ball $B(x,r)$

$$l_\varphi(x,r) = \sup_{y\in S(x,r)} \frac{|\varphi(y)-\varphi(x)|}{|y-x|}$$

and the local Lipschitz constant of a homeomorphism φ at the point x

$$l_\varphi(x) = \limsup_{r\to 0} l_\varphi(x,r).$$

If the mapping φ is differentiable at a point x, then $l_\varphi(x) = \mid \varphi'(x) \mid$.

The following result for the change of variable problem is typical for this paper:

THEOREM 1.2 (Change of variable theorem for $p \geq n$): *Let $n \leq p < \infty$. A homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon L_p^1(G') \to L_p^1(G)$ iff for every point $x \in G$*

$$\limsup_{r\to 0} \frac{l_\varphi^p(x,r)}{\varphi_v(x,r)} < K \tag{1}$$

where the constant K does not depend on x.

For $p = n$, the inequality (1) is equivalent to the classical definitions of quasiconformal homeomorphisms (see, for example, [14] or [12]). We use inequality (1) to define new classes of homeomorphisms (p-quasiconformal homeomorphisms).

Let $1 \leq p < \infty$. A homeomorphism $\varphi\colon G \to G'$ is called a **p-quasiconformal homeomorphism** at x if

$$\limsup_{r\to 0} \frac{l_\varphi^p(x,r)}{\varphi_v(x,r)} < \infty$$

and the quantity

$$K_p(x,\varphi) = \limsup_{r\to 0} \frac{l_\varphi^p(x,r)}{\varphi_v(x,r)}$$

is a p-dilatation of φ at x or, in short, the p-dilatation at x.

If

$$K_p(\varphi) = \sup_{x\in G} K_p(x,\varphi) < \infty,$$

then the homeomorphism φ is **p-quasiconformal** and $K_p(\varphi)$ is **a p-dilatation** of φ. For $p = n$, the class of p-quasiconformal homeomorphisms coincides with the class of quasiconformal homeomorphisms (see, for example, [14], [12]). The definition of p-quasiconformal homeomorphism for $p = n$ is similar to one of the classical metrical definition of quasiconformal mappings [3].

Let us denote the set of all p-quasiconformal homeomorphisms $\varphi\colon G \to G'$ by $\mathrm{QC}_p(G,G')$. If $G = G'$ we will use the notation $\mathrm{QC}_p(G)$. By Theorem 1.2 a homeomorphism φ induces a change of variable operator between the spaces $L_p^1(G)$ and $L_p^1(G')$ $(p \geq n)$ iff $\varphi \in \mathrm{QC}_p(G,G')$.

For $p < n$ the condition (1) of Theorem 1.2 is not necessary. Therefore we introduce some new classes of homeomorphisms (almost p-quasiconformal homeomorphisms). These classes give us necessary and sufficient conditions for $n-1 < p < n$.

Let $q > 1$. The quantity

$$Q_{p,q}(x,\varphi) = \limsup_{r\to 0} \frac{l_\varphi^p(x,r)}{\varphi_v(x,qr)}$$

is a (q,p)-dilatation of the homeomorphism φ at x or, in short, the (q,p)-dilatation at x. If $q = 1$ the definition is equivalent to the definition of p-quasiconformality. The quantity

$$Q_{p,q}(\varphi) \stackrel{\mathrm{def}}{=} \sup_{x\in G} Q_{p,q}(x,\varphi)$$

is a (q,p)-dilatation of the homeomorphism φ.

If there exists $q > 1$ such that

$$Q_{p,q}(\varphi) = \sup_{x\in G} Q_{p,q}(x,\varphi) < \infty,$$

then homeomorphism $\varphi\colon G \to G'$ is ***ap*-quasiconformal** (almost p-quasiconformal). Let us denote the set of all ap-quasiconformal homeomorphisms $G \to G'$ by $\mathrm{QC}_{ap}(G,G')$. If $G = G'$ we will use the notation $\mathrm{QC}_{ap}(G)$.

THEOREM 1.3 (Change of variable theorem for $n-1 < p < n$): *Let $n-1 < p < n$. A homeomorphism φ: $G \to G'$ induces the bounded operator φ^*: $L^1_p(G') \to L^1_p(G)$ iff φ is an ap-quasiconformal homeomorphism.*

If $p \geq n$ classes of *ap*-quasiconformal homeomorphisms and *p*-quasiconformal homeomorphisms coincide. For $p < n$ the classes are different and $\mathrm{QC}_p(G, G') \subset \mathrm{QC}_{ap}(G, G')$. Examples of *ap*-quasiconformal but not *p*-quasiconformal homeomorphisms can be found in Example 6 (section 4).

For $p < n - 1$ the condition of Theorem 1.3 is not necessary. For this case we do not know **a geometrical description** of the corresponding class of homeomorphisms. Necessary and sufficient analytical conditions are the following:

THEOREM 1.4 (Change of variable theorem for $1 \leq p \leq n-1$): *Let $1 \leq p \leq n-1$. A homeomorphism φ: $G \to G'$ induces the bounded operator φ^*: $L^1_p(G') \to L^1_p(G)$ iff $\varphi \in W^1_{1,\mathrm{loc}}(G)$ and almost everywhere*

$$\Big(\sum_{i,j=1}^{n} \big(\frac{\partial \varphi_i}{\partial x_j}(x)^2\big)\Big)^{p/2} \leq K\varphi_v^{'}(x)$$

where the constant K does not depend on x.

Remarks: (1) The change of variable problem had been studied in terms of multipliers of Sobolev spaces in [10]. (2) Theorem 1.4 for spaces $W^1_p(R^n)$ was obtained in [15] with additional restrictions. The change of variable problem for the case φ^*: $L^1_p(G') \to L^1_q(G)$ ($q < p$) had been studied in [18]. However, the arguments in [15] and [18] have gaps.

Let us mention some properties of *p*-quasiconformal homeomorphism which will be proved in Section 2:

(1) a *p*-quasiconformal homeomorphism is differentiable almost everywhere;

(2) for $p > 1$, a *p*-quasiconformal homeomorphism is absolutely continuous in the Tonelly sense (ACL-property);

(3) the coordinate functions of a *p*-quasiconformal homeomorphism belong to the Sobolev space $W^1_{p,loc}$;

(4) for $p > n$, a *p*-quasiconformal homeomorphism is locally Lipschitz;

(5) for $p < n$, a *p*-quasiconformal homeomorphism φ "does not compress" volume, i.e. for every compact set B, $|\varphi(B)|\,/|B| > \alpha^2 > 0$;

(6) let G, G', G'' be domains in R^n, $p \geq n$, $\varphi \in \mathrm{QC}_p(G, G')$ and $\psi \in \mathrm{QC}_p(G', G")$; then $\psi \circ \varphi \in \mathrm{QC}_p(G, G")$.

For $p \geq n$ the set $\mathrm{QC}_p(G)$ is a semigroup. Only for $p = n$ (for the quasiconformal case) is $\mathrm{QC}_n(G)$ a group. For $p < n$ we could not prove that $\mathrm{QC}_p(G)$ is a semigroup. If $p_1 > p > n$ or $p_1 < p < n$, then $\mathrm{QC}_p(G, G') \subset \mathrm{QC}_{p_1}(G, G')$.

The following example demonstrates that the behaviour of p-quasiconformal homeomorphisms ($p \neq n$) is essentially different from the behaviour of quasiconformal homeomorphisms. For $p \neq n$, p-quasiconformal homeomorphisms demonstrate some "anisotropic" properties as we can see from the following example.

Example 1: Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$; $\alpha = \sum_1^n \alpha_i$ where $-1 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and let $\varphi\colon B(0,1) \to R^n$ be the following:

$$\varphi(x) = (x_1 \cdot |x|^{\alpha_1}, x_2 \cdot |x|^{\alpha_2}, \ldots, x_n \cdot |x|^{\alpha_n}).$$

For $x \neq 0$

$$1 + \sum_{i=1}^n \alpha_i x_i^2 |x|^{-2} > \min_i(\alpha_i + 1).$$

Hence for $x \neq 0$

$$\det I(x, \varphi) = |x|^\alpha (1 + \sum_{i=1}^n \alpha_i x_i^2 |x|^{-2}) \cong |x|^\alpha$$

and $l_\varphi(x) \leq \mathrm{const} \cdot |x|^{\alpha_1}$. So the homeomorphism φ is p-quasiconformal in the set $R^n \setminus \{0\}$ for all p such that $p \cdot \alpha_1 \geq \alpha$. For $x = 0$, $l_\varphi(x, r) = r^{\alpha_1}$ and $|\varphi(B(0,r)| = \sigma_n \cdot r^{n+\alpha}$. Here σ_n is the volume of the unit ball in R^n. So the homeomorphism φ is p-quasiconformal at 0 if $p \cdot \alpha_1 \geq \alpha$.

Consider three cases: $\alpha_1 > 0$, $\alpha \cdot \alpha_1^{-1} = n$, or $\alpha_1 < 0$. (1) If $\alpha_1 > 0$ then $p \geq \alpha \cdot \alpha_1^{-1}$ and $\alpha \cdot \alpha_1^{-1} \geq n$. So the homeomorphism φ is p-quasiconformal for all $p \geq \alpha \cdot \alpha_1^{-1}$. For $p < \alpha \cdot \alpha_1^{-1}$ the homeomorphism is not p-quasiconformal. (2) If $\alpha \cdot \alpha_1^{-1} = n$ then $\alpha_1 = \alpha_i$ for all $i = 1, 2.\ldots n$ and the homeomorphism φ is quasiconformal. (3) If $\alpha \leq \alpha_1 < 0$ then φ is p-quasiconformal for all $p \in [1, \alpha\alpha_1^{-1}]$. In this case signs of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ may be different. We may choose these numbers such that $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < 0$, $\alpha_n > 0$ and $\alpha \leq \alpha_1$. For this choice the homeomorphism φ is p-quasiconformal for $1 \leq p < \alpha\alpha_1^{-1} < n - 1$ and both homeomorphisms φ, φ^{-1} do not possess the Lipschitz property.

2. The analytical properties of p-quasiconformal and ap-quasiconformal homeomorphisms

In this section we establish some differential properties of p-quasiconformal and ap-quasiconformal homeomorphisms.

From this point on we assume that G, G' are two domains in $\mathbb{R}^n$.

2.1 PRELIMINARY RESULTS. *Definition of quasiisometrical homeomorphisms.* If

$$K_\infty(\varphi) \stackrel{\text{def}}{=} \sup_{x\in G} l_\varphi(x) < \infty$$

and

$$K_\infty(\varphi^{-1}) \stackrel{\text{def}}{=} \sup_{y\in G'} l_{\varphi^{-1}}(x) < \infty,$$

then a homeomorphism $\varphi\colon G \to G'$ is **quasiisometrical** and the quantity $KI(\varphi) = \max\{K_\infty(\varphi), K_\infty(\varphi^{-1})\}$ is **a metrical dilatation** of φ.

Let us denote $\mathrm{QI}(G, G')$ the group of all quasiisometrical homeomorphisms $\varphi\colon G \to G'$.

PROPOSITION 2.1: *For all $1 \le p < \infty$, $\mathrm{QI}(G, G') \subset \mathrm{QC}_p(G, G')$ and $K_p(\varphi) \le KI^{p+n}(\varphi)$.*

Proof: It is evident that for a quasiisometrical homeomorphism $\varphi\colon G \to G'$

$$KI^{-n}(\varphi) \le \limsup_{r\to 0} \varphi_v(x, r) \le KI^n(\varphi),$$

$$KI^{-1}(\varphi) \le l_\varphi(x) \le KI(\varphi)$$

for all $x \in G$. Therefore

$$K_p(\varphi) \le KI^{n+p}(\varphi)$$

and φ is a p-quasiconformal homeomorphism. ∎

Suppose that $\varphi\colon G \to G'$ is a homeomorphism. Then φ maps every Borel set $A \subset G$ onto a Borel set. The quantity $\mu_\varphi(A) \stackrel{\text{def}}{=} |\varphi(A)|$ is regular outer measure (in the Borel sense), i.e. all open sets are μ_φ-measurable and for each set $A \subset G$ there is a Borel set $B \supset A$ such that $\mu_\varphi(A) = \mu_\varphi(B)$. Indeed, for each set $A \subset G$ there is a Borel set $C \subset G'$ such that $C \supset \varphi(A)$ and $|\varphi(A)| = |C|$. Hence $\varphi^{-1}(C)$ is the Borel set, $\varphi^{-1}(C) \supset A$ and

$$\mu_\varphi(\varphi^{-1}(C)) = |C| = |\varphi(A)| = \mu_\varphi(A).$$

From the Lebesgue theorem, one can obtain the following Poposition; see e.g. [14] Theorem 2.4.2.

PROPOSITION 2.2:

(1) $\varphi'_v(x) < \infty$ *almost everywhere;*
(2) $\varphi'_v(x)$ *is a measurable function;*
(3) *for each measurable set* $A \subset G$, $|\varphi(A)| \geq \int_A \varphi'_v(x)dx$;
(4) *if the homeomorphism* φ *is differentiable at* x *and* $I(x, \varphi)$ *is the Jacoby matrix of* f *at* x, *then*

$$\varphi'_v(x) = |\det I(x, \varphi)| \stackrel{\text{def}}{=} |I(x, \varphi)|.$$

Let us present a transparent geometric interpretation of the notion of p-quasiconformality for differentiable case.

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. There are two orthogonal bases $(\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n)$, $(\vec{g}_1, \vec{g}_2, \ldots \vec{g}_n)$ and positive numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that $\varphi(\vec{e}_i) = \lambda_i \vec{g}_i$ for $i = 1, 2, \ldots, n$ (see, for example, [12]). The vectors $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ are the eigenvectors for the symmetric linear isomorphism $\varphi^T \circ \varphi$ and the numbers $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the corresponding eigenvalues.*

If S^{n-1} is a unit sphere in R^n, then for the basis $(\vec{g}_1, \vec{g}_2, \ldots, \vec{g}_n)$ the set $\varphi(S^{n-1})$ is the surface

$$\frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \cdots + \frac{x_n^2}{\lambda_n^2} = 1,$$

$l_\varphi(0) = \lambda_n$, $|I(x, \varphi)| = \lambda_1 \lambda_2 \cdots \lambda_n$ and $K_p(\varphi) = \lambda_n^{p-1}/(\lambda_1 \lambda_2 \cdots \lambda_{n-1})$.

For a diffeomorphism $\varphi\colon G \to G'$ the positive numbers $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ correspond to the linear isomorphism $d\varphi(x)$. Then $l_\varphi(x) = \lambda_n(x)$, $|I(x, \varphi)| = \lambda_1(x)\lambda_2(x) \ldots \lambda_n(x)$ and the p-dilatation has the folowing simple form:

$$K_p(x, \varphi) = \lambda_n^{p-1}(x)/(\lambda_1(x)\lambda_2(x) \cdots \lambda_{n-1}(x)).$$

2.2 Differentiable properties of p-quasiconformal homeomorphisms

PROPOSITION 2.3: *A p-quasiconformal (ap-quasiconformal) homeomorphism* $\varphi\colon G \to G'$ *is differentiable almost everywhere (i.e the differential* $d\varphi(x)$ *exists a.e.).*

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* Here φ^T is the conjugate linear isomorphism.

Proof: Let $\varphi \in \mathrm{QC}_p(G,G')$. By Proposition 2.2, $\varphi_v'(x) < \infty$ a.e. By the definition of a p-quasiconformal homeomorphism $l_\varphi^p(x) \leq K_p(\varphi) \cdot \varphi_v'(x)$ a.e. So $l_\varphi(x) < \infty$ a.e. Recall the Rademacher–Stepanov theorem (see, for example, [1] Theorem 3.1.9):

Let a set B be open, $A \subset B \subset R^n$, $\varphi\colon B \to R^m$ and

$$\limsup_{y\to x} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} < \infty$$

for all $x \in A$. Then the mapping φ is differentiable at almost all points in A.

By this theorem a p-quasiconformal homeomorphism is differentiable a.e. For ap-quasiconformal homeomorphisms the proof is the same. ∎

Let $R_i^{n-1} \stackrel{\text{def}}{=} \{x \in R^n |\ x_i = 0\}$, P_i be the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}_i^{n-1}$ and $Q \stackrel{\text{def}}{=} \{x \in R^n |\ a_j \leq x_j \leq a_j + d,\ d > 0,\ j = 1,2,\dots n\}$ be a closed cube. If a mapping $\varphi\colon Q \to \mathbb{R}^n$ is continuous and absolutely continuous on almost every line segment in Q, parallel to coordinate axes, then φ is ACL on Q (absolutely continuous on lines).

If U is an open set in $\mathbb{R}^n$ and a mapping $\varphi\colon U \to \mathbb{R}^n$ is ACL on every closed cube $Q \subset U$, then φ is ACL.

The theorem below is an extension to the p-quasiconformal case of the corresponding results for quasiconformal homeomorphisms [2] and [14] section 31.2.

THEOREM 2.4: *Let $p > 1$. A p-quasiconformal (ap-quasiconformal) homeomorphism $\varphi\colon G \to G'$ is ACL.*

The proof is essentially the same as that presented in [14].

Remark 1: The case $p = 1$ is exceptional. A 1-quasiconformal homeomorphism is not necessarily ACL, as can be seen from the following example:

Example 2: Let $\nu\colon R \to R$ be a continuous increasing function. Suppose that ν is not an absolutely continuous function. The mapping $\varphi(x) \stackrel{\text{def}}{=} (x_1 + \nu(x_1), x_2, \dots, x_n)$ is a 1-quasiconformal homeomorphism. The homeomorphism φ is not ACL. So Theorem 2.4 is not correct for the case $p = 1$.

COROLLARY 2.5: *Let $p > 1$. Then a p-quasiconformal (ap-quasiconformal) homeomorphism $\varphi\colon C \to G'$ belongs to $W^1_{p,loc}(G)$.*

Proof: By Proposition 2.3, φ is differentiable a.e., i.e. there is a measurable set $E \subset G$ such that $|G - E| = 0$ and φ is differentiable in E. Let $B(x_0, r) \subset G$ be an open ball, $B_1 = E \cap B(x_0, r)$. For every point $x \in B_1$ we have $|\nabla(\varphi_i(x))| \leq l_\varphi(x)$ for $i = 1, 2, \ldots, n$. By the definitions and Proposition 2.2

$$\int_{B(x_0,r)} |\nabla(\varphi_i(x))|^p dx = \int_{B_1} |\nabla(\varphi_i(x))|^p dx$$
$$\leq K_p(\varphi) \int_{B_1} \varphi'_v(x) dx \leq K_p(\varphi)|\varphi(B(x_0, r))| < \infty.$$

So $\varphi \subset W^1_{p,loc}(G)$, because φ is ACL. ∎

2.3 Metric properties of p-quasiconformal homeomorphisms

Proposition 2.6: *If $p > n$ and $\varphi \in \mathrm{QC}_p(G, G')$, then*

$$l_\varphi(x) \leq [K_p(\varphi)]^{1/(p-n)}$$

for all $x \in G$.

Corollary 2.7: *If $p > n$ and $\varphi \in \mathrm{QC}_p(G, G')$, then φ is a locally Lipschitz homeomorphism.*

Proof of the proposition: Fix a point $x \in G$ and $\varepsilon > 0$. By the definition of the p-quasiconformality there is r_0 such that, for all $0 < r < r_0$,

$$l^p_\varphi(x, r) \leq [K_p(\varphi) + \varepsilon] \cdot \varphi_v(x, r) \leq [K_p(\varphi) + \varepsilon] l^n_\varphi(x, r).$$

So $l_\varphi(x, r) \leq [K_p(\varphi) + \varepsilon]^{1/(p-n)}$.

Letting $r \to 0$ and $\varepsilon \to 0$ we obtain the desire result. ∎

Proposition 2.8: *If $p < n$ and $\varphi: G \to G'$ is a p-quasiconformal homeomorphism, then for every measurable set B*

$$\frac{|\varphi(B)|}{|B|} \geq [K_p(\varphi)]^{n/(p-n)}.$$

Proof: Fix a point $x \in G$ and $\varepsilon > 0$. By the definition of a p-quasiconformal homeomorphism

$$[\varphi'_v]^{p/n} \leq (l_\varphi(x))^p \leq [K_p(\varphi)]\varphi'_v.$$

Hence

$$\varphi'_v(x) \geq [K_p(\varphi)]^{n/(p-n)}.$$

By Proposition 2.2

$$|B|[K_p(\varphi)]^{n/(p-n)} \leq \int_B \varphi'_v(x) dx \leq |\varphi(B)|. \quad ∎$$

PROPOSITION 2.9: *If $p < n$ and $\varphi\colon G \to G'$ is a p-quasiconformal homeomorphism, then $l_\varphi(x) \geq [K_p(\varphi)]^{1/(p-n)}$.*

Proof: Fix a point $x \in G$ and $\varepsilon > 0$. By the definition of a p-quasiconformal homeomorphism there is an r_0 such that, for all $0 < r < r_0$,

$$l_\varphi^p(x,r) \leq [K_p(\varphi)+\varepsilon]\varphi_v(x,r) \leq [K_p(\varphi)+\varepsilon]\cdot l_\varphi^n(x,r).$$

So $l_\varphi(x) \geq [K_p(\varphi)]^{1/(p-n)}$. ∎

2.4 Relationship between classes of p-quasiconformal homeomorphisms for different p

THEOREM 2.10: *If $p_1 > p > n$ or $p_1 < p < n$, then $\mathrm{QC}_p(G,G') \subset \mathrm{QC}_{p_1}(G,G')$.*

Proof: Let $\varphi \in \mathrm{QC}_p(G,G')$, $p > 1$, $p \neq n$. Fix $x \in G$ and $\varepsilon > 0$. There exists an r_0 such that, for all $0 < r < r_0$, we have: (1) By the definition of a p-quasiconformal homeomorphism

$$l_\varphi^p(x,r) \leq [K_p(\varphi)+\varepsilon]\varphi_v(x,r).$$

(2) By Proposition 2.6, for $p > n$

$$l_\varphi(x,r) \leq [K_p(\varphi)+\varepsilon]^{1/(p-n)}$$

or by Proposition 2.9, for $p < n$

$$l_\varphi(x,r) \geq [K_p(\varphi)+\varepsilon]^{1/(p-n)}.$$

In either case the inequality

$$l_\varphi^{p_1-p}(x,r) \leq [K_p(\varphi)+\varepsilon]^{(p_1-p)/(p-n)}$$

is valid.

These two inequalities imply

$$l_\varphi^{p_1}(x,r) = l_\varphi^p(x,r)l_\varphi^{p_1-p}(x,r) \leq [K_p(\varphi)+\varepsilon]^{(p_1-n)/(p-n)}\varphi_v(x).$$

So $K_{p_1}(\varphi) \leq [K_p(\varphi)]^{(p_1-n)/(p-n)}$ and $\varphi \in \mathrm{QC}_{p_1}(G,G')$. ∎

THEOREM 2.11: *The set $\mathrm{QC}_n(G,G')$ coincides with the set of all quasiconformal homeomorphisms $\mathrm{QC}(G,G')$.*

See [14] (section 22.3) or [12] .

PROPOSITION 2.12: *If* $\varphi \in \mathrm{QC}(G,G') \cap \mathrm{QC}_\infty(G,G')$ *then* $\varphi \in \mathrm{QC}_p(G,G')$ *for all* $p > n$.

Proof: Let $\varphi \in \mathrm{QC}(G,G') \cap \mathrm{QC}_\infty(G,G')$. Fix a point $x \in G$ and $\varepsilon > 0$. By the definitions of quasiconformal homeomorphisms there is an r_0 such that, for all $0 < r < r_0$, $l_\varphi(x,r) \le K_\infty(\varphi)$ and

$$l^n_\varphi(x,r) \le [K_n(\varphi) + \varepsilon]\varphi_v(x,r).$$

For $p > n$

$$l^p_\varphi(x,r) \le [K_\infty(\varphi)+\varepsilon]^{p-n} l^n_\varphi(x,r) \le [K_\infty(\varphi)+\varepsilon]^{p-n}[K_n(\varphi)+\varepsilon]\varphi_v(x,r).$$

So $K_p(\varphi) \le K_\infty(\varphi)^{p-n} K_n(\varphi)$ and $\varphi \in \mathrm{QC}_p(G,G')$. ∎

PROPOSITION 2.13: *If* $\varphi \in \mathrm{QC}(G,G')$ *and* $\varphi^{-1} \in \mathrm{QC}_\infty(G,G')$, *then* $\varphi \in \mathrm{QC}_p(G, G')$ *for all* $p < n$.

Proof: Let $\varphi \in \mathrm{QC}(G,G')$ and $\varphi^{-1} \in \mathrm{QC}_\infty(G,G')$. Fix a point $x \in G$ and $\varepsilon > 0$. By the definitions there is r_0 such that, for all $0 < r < r_0$, $l^n_\varphi(x,r) \le K_n(\varphi)\cdot\varphi_v(x,r)$ and $l_\varphi(x) \ge [K_\infty(\varphi^{-1})]^{-1}$. From these two inequalities we obtain

$$\begin{aligned} l^p_\varphi(x,r) \le& [\{K_\infty(\varphi^{-1})\}^{-1} - \varepsilon]^{p-n} l^n_\varphi(x,r) \\ \le& [\{K_\infty(\varphi^{-1})\}^{-1} - \varepsilon]^{p-n}[K_n(\varphi)+\varepsilon]\varphi_v(x,r). \end{aligned}$$

So $\varphi \in \mathrm{QC}_p(G,G')$ for all $p < n$. ∎

3. Change of variable operators for Sobolev spaces (weak version)

In this section we present some new results about the change of variable operators for Sobolev spaces and establish some additional analytical properties of p-quasiconformal homeomorphisms.

3.1 CHANGE OF VARIABLE THEOREMS FOR L^1_p-SPACES (SUFFICIENT CONDITIONS). Let $\varphi: G \to G'$ be a homeomorphism and

$$K_\infty(\varphi) \overset{\text{def}}{=} \sup_{x\in G} l_\varphi(x).$$

Definition: If $K_{\infty}(\varphi) < \infty$, then a homeomorphism φ is **quasilipschitz** (or ∞-quasiconformal) and the quantity $K_{\infty}(\varphi)$ is a **quasilipschitz dilatation** of φ. Let us denote the set of all the quasilipschitz homeomorphisms $\mathrm{QC}_{\infty}(G, G')$.

It turns out that the p-quasiconformal homeomorphisms induce change of variable operators for Sobolev spaces L^1_p. The proof of this fact is presented in Theorem 3.1.

THEOREM 3.1: *Let $1 < p \leq \infty$. A p-quasiconformal homeomorphism $\varphi: G \to G'$ induces the bounded operator $\varphi^*: L^1_p(G') \to L^1_p(G)$.*

For $1 < p < \infty$ we have $||\varphi^*|| = K_p^{1/p}(\varphi)$. For $p = \infty$ we have $||\varphi^*|| = K_{\infty}(\varphi)$.

Proof: 1. For $p = n$ the theorem follows from Theorem 1.1.

2. Let $p > n$ and let $u \in L^1_p(G')$ be a smooth function. The p-quasiconformal homeomorphism φ is locally Lipschitz (Corollary 2.7). Hence $u \circ \varphi$ is locally Lipschitz also and $\nabla(u \circ \varphi)(x)$ exist almost everywhere. By the definition of a p-quasiconformal homeomorphism

$$\int_G |\nabla_x(u \circ \varphi)(x)|^p dx \leq \int_G |\nabla_y u(\varphi(x))|^p \cdot [l_{\varphi}(x)]^p dx$$
$$\leq K_p(\varphi) \cdot \int_G |\nabla_y u(\varphi(x))|^p \varphi_v'(x) dx.$$

By Proposition 2.3, φ is differentiable a.e. So $\varphi_v'(x) = I(x, \varphi)$ a.e. Therefore (see, for example, Theorem 3.2.3 [1])

$$\int_G |\nabla_y u(\varphi(x))|^p \varphi_v'(x) dx = \int_G |\nabla_y u(\varphi(x))|^p |I(x, \varphi)| dx$$
$$= \int_{G'} |\nabla_y u(y)|^p dy.$$

Hence

$$\int_G |\nabla_x(u \circ \varphi)(x)|^p dx \leq K_p(\varphi) \int_{G'} |\nabla_y u(y)|^p dy$$

or

$$||\varphi^* u||_{1,p} \leq K_p^{1/p}(\varphi) ||u||_{1,p}. \tag{2}$$

Thus (2) is proved for $u \in C_{\infty}(G) \cap L^1_p(G')$. Because the smooth functions are dense in the space $L^1_p(G')$ and the operator φ^* is bounded on this dense set, we can extend the inequality (2) for all $u \in L^1_p(G')$. So the homeomorphism φ induces the bounded operator φ^*.

3. Let $1 < p < n$ and let $u \in L^1_p(G')$ be smooth. The p-quasiconformal homeomorphism $\varphi \in W^1_{p,\mathrm{loc}}(G)$ (Corollary 2.5). So $u \circ \varphi$ is ACL and it has first partial derivatives a.e.

Consider the outer measure $\mu(A) \stackrel{\mathrm{def}}{=} \int_A [l_\varphi(x)]^p dx$ for every $A \subset G$. By 2.4.10 [1] every (Lebesgue) measurable set is μ-measurable and for every (Lebesgue) measurable function u

$$\int_A u\, d\mu = \int_A u(x)[l_\varphi(x)]^p dx. \tag{3}$$

By the definition of p-quasiconformal homeomorphisms for every measurable set $A \subset G$

$$\mu(A) = \int_A [l_\varphi(x)]^p dx \le K_p(\varphi) \int_A \varphi'_v(x) dx \le K_p(\varphi) \cdot |\varphi(A)|. \tag{4}$$

For the outer measure

$$\varphi_\#(\mu)(B) \stackrel{\mathrm{def}}{=} \mu(\varphi^{-1}(B))$$

the inequality (4) implies

$$\varphi_\#(\mu)(B) \stackrel{\mathrm{def}}{=} \mu(\varphi^{-1}(B)) \le K_p(\varphi)|B|. \tag{5}$$

From (3) we obtain

$$\begin{aligned} ||\varphi^* u||^p_{1,p} &= \int_G |\nabla_x (u \circ \varphi)(x)|^p dx \\ &\le \int_G |\nabla_y u(\varphi(x))|^p \cdot [l_\varphi(x)]^p dx = \int_G |\nabla_y u(\varphi(x))|^p d\mu. \end{aligned} \tag{6}$$

Because u is smooth the function $\nabla_y u$ exists everywere in G'. The function $\nabla_y u(\varphi(x))$ is μ-measurable. Hence by Theorem 2.4.18 [1]

$$\int_G |\nabla_y u(\varphi(x))|^p d\mu = \int_{G'} |\nabla_y u(y)|^p d\varphi_\#(\mu). \tag{7}$$

Now inequality (5) implies

$$\int_{G'} |\nabla_y u(y)|^p d\varphi_\#(\mu) \le K_p(\varphi) \int_{G'} |\nabla_y u(y)|^p dy. \tag{8}$$

So from inequalities (6), (7) and (8) we obtain

$$||\varphi^* u||^p_{1,p} \le K_p(\varphi) ||u||^p_{1,p}.$$

The end of the proof is the same as for the case 2.

4. Let $p = \infty$ and $\varphi \in \mathrm{QC}_\infty(G, G')$. By the definition of quasilipschitz homeomorphisms $\sup_{x\in G} l_\varphi(x) \stackrel{\text{def}}{=} K_\infty(\varphi) < \infty$. For every function $u \in L^1_\infty(G)$ we obtain

$$\begin{aligned} ||\varphi^* u||_{L^1_\infty(G)} &\stackrel{\text{def}}{=} ||\nabla_x(u \circ \varphi)||_{L_\infty(G)} \\ &\leq ||\nabla_y(u(\varphi(x))) \cdot l_\varphi(x)||_{L_\infty(G)} \end{aligned}$$

and

$$K_\infty(\varphi)||\nabla_y(u)||_{L_\infty G'} \stackrel{\text{def}}{=} K_\infty(\varphi)||u||_{L^1_\infty(G')}. \qquad \blacksquare$$

Remarks: 1. For *ap*-quasiconformal homeomorphisms Theorem 2.4 is also correct. The proof is the same.

2. The case $p = 1$ is exceptional. A 1-quasiconformal homeomorphism is not necessarily ACL (see Remark 1 to Theorem 2.4). If a homeomorphism does not possess the ACL-property it does not induce a change of variable operator for the corresponding Sobolev spaces.

For $p = 1$, Theorem 3.1 is correct with an additional restriction that φ is a 1-quasiconformal homeomorphism with the ACL-property.

3.2 CHANGE OF VARIABLE THEOREMS FOR W^1_p-SPACES (SUFFICIENT CONDITIONS). Let us recall the definition of p-Poincaré domains. Let G be a bounded domain. For an integrable function u on G the number u_G is the average value of u, i.e.

$$u_G = |G|^{-1} \int_G u\, dx.$$

The p-Poincaré constant $1 < p < \infty$ of G is given by the equality

$$\mathcal{P}(G) = \sup_{u \in W^1_p(G),\, u \neq \text{const}} \frac{||u - u_G||_{L_p(G)}}{||u||_{L^1_p(G)}}.$$

The domain G is said to be a p-**Poincaré domain** if $\mathcal{P}(G)$ is finite. One can find detailed information about Poincaré domains in the papers [6], [7], [10], and [13].

THEOREM 3.2: *Let G be a p-Poincaré domain, $1 < p < \infty$ and let a homeomorphism $\varphi\colon G \to G'$ be p-quasiconformal. Then for every function $u \in W^1_p(G')$ the following inequality*

$$||\varphi^*(u) - \varphi^*(u)_G||_{W^1_p(G')} \leq H \cdot ||u||_{W^1_p(G')}$$

is correct. Here the constant H depends only on the p-dilatation $K_p(\varphi)$ and the Poincaré constant $\mathcal{P}(G)$.

Proof: By the Poincaré inequality and Theorem 3.1

$$\begin{aligned}&\|\varphi^*(u) - \varphi^*(u)_G\|_{W_p^1(G)} \le \mathcal{P}(G)\|\varphi^*(u)\|_{L_p^1(G)}\\ &\le \mathcal{P}(G)K_p^{1|p}(\varphi)\|u\|_{L_p^1(G')} \le \mathcal{P}(G)K_p^{1|p}(\varphi)\|u\|_{W_p^1(G')}.\end{aligned}$$ ∎

Remark: The class of p-Poincaré domains includes the class of bounded domains with Lipschitz boundaries.

THEOREM 3.3: *Let $1 < p \le n$. A p-quasiconformal homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon W_p^1(G') \to W_p^1(G)$.*

Proof: By Proposition 2.8, there is a number C such that $|\varphi(A)| \ge C|A|$ for each measurable set A and $\varphi'(x) \ge C$ a.e. Hence

$$\begin{aligned}\|u \circ \varphi\|^p_{L_p(x)} &= \int_G |u \circ \varphi|^p(x)dx\\ &\le \frac{1}{C}\int_G |u \circ \varphi|^p(x)\varphi_v'(x)dx \le \frac{1}{C}\int_{G'} |u(y)|^p dy = \frac{1}{C}\|u\|_{L_p(G')}.\end{aligned}$$

So the statement of the theorem follows from Theorem 3.2. ∎

3.3 WEAKLY p-QUASICONFORMAL HOMEOMORPHISMS. For technical reasons we need to introduce new classes of homeomorphisms.

Let $\varphi\colon G \to G'$ be a homeomorphism and let $\varphi \in W^1_{1,loc}(G)$. Define

$$\bar{l}_\varphi(x) = \max_{\alpha=(\alpha_1,\alpha_2,\dots\alpha_n),\ |\alpha|=1} \sum_i \alpha_i|\nabla\varphi_i(x)|.$$

The quantity

$$K_p^1(x,\varphi) \overset{\text{def}}{=} \inf\{C\bar{l}_\varphi(x) \le C\varphi_v'(x)\}$$

is a weak p-dilatation of the homeomorphism φ at x. If $\bar{l}_\varphi(x) > 0$ and $\varphi_v'(x) = 0$, then $K_p^1(x,\varphi) = \infty$. The weak p-dilatation $K_p^1(x,\varphi)$ exists a.e. because $\bar{l}_\varphi(x)$ and the volume derivative $\varphi_v'(x)$ exist a.e. For differentiable mappings $\varphi_v'(x) = |\det I(x,\varphi)|$.

If $\varphi \in W^1_{1,loc}(G)$ and

$$K_p^1(\varphi) = |\, K_p^1(x,\varphi)|_{L_\infty(G)} < \infty,$$

then the homeomorphism $\varphi\colon G \to G'$ is a **weakly p-quasiconformal** and the quantity $K_p^1(\varphi)$ is **a weak p-dilatation** of the homeomorphism φ.

For $p > 1$ an ap-quasiconformal homeomorphism is a weakly p-quasiconformal (Corollary 2.5 and the definition of ap-quasiconformality). As it will be seen later, if $p > n-1$ then the classes of the ap-quasiconformal homeomorphisms and the weakly p-quasiconformal homeomorphisms coincide (Theorem 4.10). For $p < n-1$ the classes are different. In Section 4, Example 5 we demonstrate an example of a weakly p-quasiconformal which is not an ap-quasiconformal homeomorphism.

The following theorem is in fact a version of Theorem 3.1.

THEOREM 3.4: *Let $1 \le p < \infty$. A weakly p-quasiconformal homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon L_p^1(G) \to L_p^1(G')$.*

The proof is almost the same as the proof of Theorem 3.1 . Instead of $l_{\varphi(x)}$ it is necessary to use $\bar{l}_\varphi(x)$ in the proof of Theorem 3.1. ∎

3.4 CHANGE OF VARIABLE THEOREMS (WEAK NECESSARY CONDITIONS). Let $\varphi\colon G \to G'$ be a continuous mapping. Fix some domain $\Omega \subset \mathbb{R}^k$. Suppose that $\{K_y\}_{y\in\Omega}$ is a family of compact sets such that $K_y \cap K_{y_1} = \emptyset$ if $y_1 \neq y$ and $K_y \subset G'$ for all $y \in \Omega$.

LEMMA 3.5: *$|\varphi^{-1}(K_y)| = 0$ for all $y \in \Omega$ except possibly a countable subset of Ω.*

Proof: Let $A_s = \{y \in \Omega\colon |\varphi^{-1}(K_y)| > 1/s\}$, $s = 1, 2, \dots$. It is obvious that every set A_s is at most a countable set. So $A = \{y \in \Omega\colon |\varphi^{-1}(K_y)| > 0\}$ is at most countable too.

Now we turn to the investigation of the necessary analytic conditions for the change of variable problem. The first step in this direction is the following:

THEOREM 3.6: *If a homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon L_p^1(G') \to L_p^1(G)$ for $p \ge 1$, then $\varphi \in W^1_{p,\mathrm{loc}}(G)$ and there is a number $Q_p(\varphi)$ such that*

$$|\nabla\varphi_i(x)|^p \le Q_p(\varphi)\varphi'_v(x)$$

for almost all $x \in G$ and for all $i = 1, 2, \dots, n$.

Proof: The proof is based on construction of a function ψ whose graph is of a "saw-type" with adjacent orthogonally meeting segments. The action of φ^*

on this particular function defines the differentiable properties of the homeomorphism φ.

By Proposition 2.2, $\varphi'_v(x) < \infty$ a.e. Fix $x_0 \in G$ such that $\varphi'_v(x_0) < \infty$. Let $\varepsilon > 0$, $k > 1$ be arbitrary. There exists an r_0 such that, for all $0 < r \leq r_0$,

$$(9) \qquad |\varphi(B(x_0, kr))| \leq (\varphi'_v(x) + \varepsilon)|B(x_0, kr)| \leq (\varphi'_v(x) + \varepsilon)k^n|B(x_0, r)|.$$

Let $M = (\varphi'_v(x)+\varepsilon)k^n$. Let us call a cube Q h-**regular** if all its edges are parallel to the corresponding coordinate axes, the length of the edge is h and every vertex b has the form $k_1h, k_2h, \ldots, k_nh$ where $k_1, k_2, \ldots, k_n$ are integers. If Q_1 and Q_2 are h-regular cubes $Q_1 \neq Q_2$, then $\operatorname{Int} Q_1 \cap \operatorname{Int} Q_2 = \emptyset$. Fix $r < r_0$ and choose $h > 0$ such that

$$h < \frac{1}{2n^{1/2}} \operatorname{dist}(\varphi(S(x_0, kr)), \varphi(S(x_0, r))).$$

Let Q be the union of all h-regular cubes Q_α such that $Q_\alpha \cap \varphi(B(x_0, r)) \neq \emptyset$. It is evident that

$$\varphi(B(x_0, r)) \subset Q \subset \varphi(B(x_0, kr)).$$

Fix the axis x_j. Let us denote the hyperplanes $x_j = t\,h$ by L_t. The hyperplanes L_m (m is an integer) divide $\mathbb{R}^n$ into the layers

$$Z_m = \left\{x \in R^n |\, mh < x_j < \frac{m+1}{h}\right\}.$$

Let $Q_m = Q \cap Z_m$.

For every Q_m we construct three functions: $\psi_{m,1} = x_j - mh$, $\psi_{m,2} = (m+1)h - x_j$, $\psi_{m,3} = \frac{1}{2}h - \operatorname{dist}(P_j(x), P_j(Q_m))$. Here P_j: $\mathbb{R}^n \to R_j^{n-1}$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}_j^{n-1}$. Consider the following functions $\psi_m = \max\{0, \min\{\psi_{m,1}, \psi_{m,2}, \psi_{m,3}\}\}$, $\psi = \sum_m \psi_m$, and $E = \{x \in G |\ \psi(x)$ is not differentiable at the point $x\}$. It follows from the definition of ψ that it has the following properties:

(1) $\operatorname{Supp}(\psi) \subset \varphi(B(x_0, kr))$;
(2) $\psi \in W_p^1(G)$ for all $1 \leq p \leq \infty$;
(3) ψ is differentiable almost everywhere;
(4) $|\nabla\psi(x)| = 1$ for all points $x \in \varphi(B(x_0, r)) \setminus E$;
(5) $|\nabla\psi(x)| \leq 1$ almost everywhere;
(6) $\psi(x) = \mp x_j + \text{const}$ into all components of the set $\varphi(B(x_0, r))\backslash E$.

The set $E \cap \varphi(B(x_0,r))$ belongs to a finite union of hyperplanes $L_{t_1}, L_{t_2}, \ldots, L_{t_s}$, where $2t_i$ is an integer. By Lemma 3.5 for almost all small translations τ_y parallel to the axis x_j

$$\left|\varphi^{-1}\Big(\tau_y\big(E \cap \varphi(B(x_0,r))\big)\Big)\right| = 0.$$

So we can assume that

(7) $|\varphi^{-1}(E \cap \varphi(B(x_0,r)))| = 0$.

By the assumption of the theorem, $\varphi^*(\psi) = \psi \circ \varphi \in L^1_p(G)$. By the properties 6 and 7 of the function ψ, we have $\varphi^*(\psi)(x) = \mp\varphi_j(x) + \text{const}$ for almost all $x \in B(x_0,r)$. Therefore

$$\begin{aligned}\left\{\int_{|B(x_0,r)|} |\nabla(\varphi_j(x))|^p dx\right\}^{1/p} \leq & \quad \left\{\int_G |\nabla(\psi \circ \varphi(x))|^p dx\right\}^{1/p} \\ (10) \qquad\qquad = & \quad |\,\varphi^*(\psi)||_{L^1_p(G)}.\end{aligned}$$

Since φ^* is bounded and $\operatorname{Supp}(\psi) \subset \varphi(B(x_0,kr))$

$$|\,\varphi^*||_{L^1_p(G')} \leq |\,\varphi^*|\,|\,\psi|_{\,L^1_p(B(x_0,kr))} \leq |\,\varphi^*|\left\{\int_{B(x_0,kr)} |\nabla\psi(x)|^p dx\right\}^{1|p}.$$

By the property 4 of the function ψ, $|\nabla\psi| \leq 1$ a.e. Hence

$$(11) \qquad \int_{B(x_0,kr)} |\nabla\psi(x)|^p dx \leq |\varphi(B(x_0,kr))|.$$

From (10)–(11) it follows that

$$\int_{B(x_0,r)} |\nabla\varphi_j(x)|^p dx \leq |\,\varphi^*|\,^p\big|\varphi(B(x_0,kr))\big|.$$

By (9)

$$\int_{B(x_0,r)} |\nabla\varphi_j(x)|^p dx \leq M|\varphi^*|^p\big|(B(x_0,r))\big|.$$

Hence

$$\limsup_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\nabla\varphi_j(x)|^p dx \leq M||\varphi^*|\,^p.$$

By the Lebesgue theorem and (9)

$$|\nabla\varphi_j(x_0)|^p \leq |\,\varphi^*|\,^p(\varphi_v'(x) + \varepsilon)k^n.$$

Letting at first $\varepsilon \to 0$, then $k \to 1$ we obtain

$$|\nabla\varphi_j(x_0)|^p \leq |\,\varphi^*|\,^p\varphi_v'(x)$$

for almost all x_0. ∎

Remark: By Theorem 3.1 and Theorem 3.6, a p-quasiconformal homeomorphism is a weakly p-quasiconformal for $p > 1$.

3.5 CONCLUSIONS. All previous results allow us to obtain an analytical solution of the change of variable problem and some new results useful for the geometric solution. The latter will be presented in Section 4. Theorems 3.7–3.10 summarize the previous results.

PROPOSITION 3.7: *A weakly p-quasiconformal homeomorphism belongs to the space* $W^1_{p,\mathrm{loc}}(G)$.

Proof: By Proposition 2.2, for every open ball $B(x,r)$

$$\int_{B(x,r)} |\nabla \varphi_j(x)|^p dx \le K^1_p(\varphi) \int_{B(x,r)} \varphi'_v(x) dx$$
$$\le K^1_p(\varphi) |\varphi(B(x,r))| \le \infty. \qquad \blacksquare$$

Theorem 3.2 and Theorem 3.6 imply the following results.

THEOREM 3.8: *Let* $1 \le p \le \infty$. *A homeomorphism* $\varphi\colon G \to G'$ *induces the bounded operator* $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ *iff* φ *is a weakly p-quasiconformal homeomorphism.*

Now we are able to realize the following fact which is important for applications. Namely, if a homeomorphism induces a change of variable operator in a domain, then it induces a change of variable operator in all subdomains.

THEOREM 3.9 (Localization Principle): *Let* G, G' *be domains in* R^n *and* $V \subset G$ *be a subdomain of the domain* G. *If a homeomorphism* $\varphi\colon G \to G'$ *induces the bounded operator* $\varphi^*\colon L^1_p(G') \to L^1_p(G)$, *then* $\varphi|V$ *induces the bounded operator* $(\varphi|V)^*\colon L^1_p(\varphi(V)) \to L^1_p(V)$.

The next theorem establishes the fact that weakly p-quasiconformal homeomorphisms form a semigroup.

THEOREM 3.10: *Let* G, G', G'' *be domains in* R^n. *If* $\varphi\colon G \to G'$ *and* $\psi\colon G' \to G''$ *are weakly p-quasiconformal homeomorphisms, then the homeomorphism* $\psi \circ \varphi$ *is a weakly p-quasiconformal homeomorphism.*

Proof: By Theorem 3.8, the homeomorphisms φ, ψ induce the bounded operators $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ and $\psi^*\colon L^1_p(G'') \to L^1_p(G')$. The composition $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ is a bounded operator from $L^1_p(G'')$ to $L^1_p(G)$. By Theorem

3.8 the homeomorphism $\psi \circ \varphi$ is a weakly p-quasiconformal homeomorphism. ∎

4. Additional properties of p-quasiconformal homeomorphisms and strong versions of the change of variable theorem

In this section we prove the coincidence of the class of weakly p-quasiconformal homeomorphisms with the class of p-quasiconformal homeomorphisms for $p \geq n$ and the coincidence of the class of weakly p-quasiconformal homeomorphisms the class of ap-quasiconformal homeomorphisms for $p > n - 1$. At the end we prove the change of variable theorem for $p > n - 1$ (Theorems 1.2 and 1.3 from Introduction) and give some applications of the p-quasiconformality for the embedding theorems.

4.1 CAPACITY. Let F_0, F_1 be closed subsets of the domain G such that $F_0 \cap F_1 = \emptyset$. We define a continuous function $u \in L^1_p(G)$ to be p-**admissible** for the pair F_0, F_1 if $u \equiv 0$ on F_0, $u \equiv 1$ on F_1 and $0 \leq u(x) \leq 1$ for all $x \in G$.

The quantity

$$\mathrm{Cap}_p(F_0, F_1, G) = \inf_u |\, u|^p_{L^1_p(G)}$$

is the capacity of the pair F_0, F_1 in the space $L^1_p(G)$ (so called p-capacity). Here u is an admissible function.

If admissible functions for a pair (F_0, F_1) do not exist, then $\mathrm{Cap}_p(F_0, F_1, G) = \infty$.

PROPOSITION 4.1: *If a homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon L^1_p(G') \to L^1_p(G)$, then*

$$\mathrm{Cap}_p(F_0, F_1, G) \leq |\, \varphi^*|^p \mathrm{Cap}_p(\varphi(F_0), \varphi(F_1), G')$$

for every pair of closed sets $F_0, F_1 \subset G$.

The proof is evident.

COROLLARY 4.2: *If a homeomorphism $\varphi\colon G \to G'$ is p-quasiconformal, then*

$$\mathrm{Cap}_p(F_0, F_1, G) \leq K_p(\varphi) \mathrm{Cap}_p(\varphi(F_0), \varphi(F_1), G').$$

Proof: By Theorem 3.1, φ induces the bounded operator $\varphi^*: L_p^1(G') \to L_p^1(G)$ and $|\varphi^*|^p \le K_p(\varphi)(K_p^1(\varphi))$. ∎

Similarly, one can prove the proposition when a p-quasiconformal homeomorphism is replaced by a weakly p-quasiconformal one.

The following fact will be used in the proof of Theorem 4.4 and it is known ([4] Proposition 6.8).

PROPOSITION 4.3: *Let $B(x_0, r)$ be an open ball, $a, b \in B(x_0, r)$, $p > n$. Then*

$$\mathrm{Cap}_p(\{a\}, \{b\}, B(x_0, r)) \ge \alpha^2(n,p)|a-b|^{n-p}$$

where $\alpha^2(n,p)$ depends only on the numbers n and p.

THEOREM 4.4: *Let $p \ge n$. A weakly p-quasiconformal homeomorphism $\varphi: G \to G'$ is p-quasiconformal.*

Proof: Let a ball $B(x_0, r)$ be such that $B(x_0, 2r) \subset G$,

$$L = \sup_{z \in S(x_0, r)} |\varphi(x_0) - \varphi(z)|$$

and a point z_0 be such that $L = |\varphi(x_0) - \varphi(z_0)|$. Without loss of generality we can suppose that $\varphi(x_0) = 0$ and $\varphi(z_0) = (L, 0, \dots, 0)$. Consider the function $u(y) = y_1/L$ for $y = (y_1, y_2, \dots y_n) \in G'$. The function u is p-admissible for the pair $\{\varphi(x_0)\}$, $\{\varphi(z_0)\}$.

Let $p > n$. By the Localization Principle (Theorem 3.9) the homeomorphism $\varphi: B(x_0, r+\varepsilon) \to \varphi(B(x_0, r+\varepsilon))$ induces the bounded operator $\varphi^*: L_p^1(\varphi(B(x_0, r+\varepsilon))) \to L_p^1(B(x_0, r+\varepsilon))$. Hence Proposition 4.2 and Proposition 4.3 imply

$$\begin{aligned}\alpha^2(n,p)\cdot r^{n-p} &\le \mathrm{Cap}_p(\{x_0)\}, \{z_0\}, B(x_0, r+\varepsilon)) \\ &\le \|\varphi^*\|^p \mathrm{Cap}_p(\{\varphi(x_0)\}, \{\varphi(z_0)\}, \varphi(B(x_0, r+\varepsilon)) \\ &\le \|\varphi^*\|^p \int_{\varphi(B(x_0, r+\varepsilon))} |\nabla(u(y))|^p dy = \|\varphi^*\|^p \frac{|\varphi(B(x_0, r+\varepsilon))|}{L^p}.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\frac{l_\varphi(x,r)^p}{|\varphi_v(x,r)|} \le \frac{K_p^1(\varphi)}{\Omega_n \alpha^2(n,p)}.$$

So the homeomorphism φ is p-quasiconformal. The proof for the case $p = n$ (for quasiconformal homeomorphisms) is given in [14]. ∎

Theorems 3.8 and 4.4 imply Theorem 1.2 from the Introduction, which we may reformulate now in the new language as follows:

THEOREM 1.2: *Let $n \leq p \leq \infty$. A homeomorphism $\varphi\colon G \to G'$ induces the bounded operator $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ iff φ is a p-quasiconformal homeomorphism.*

The next theorem establishes the fact that for $p \geq n$, p-quasiconformal homeomorphisms form a semigroup.

THEOREM 4.5: *Let G, G', G'' be domains in $\mathbb{R}^n$ and let $n \leq p \leq \infty$. If $\varphi \in \mathrm{QC}_p(G,G')$ and $\psi \in \mathrm{QC}_p(G',G'')$, then the homeomorphism $\psi \circ \varphi \in \mathrm{QC}_p(G,G'')$.*

Proof: By Theorem 3.1 the homeomorphisms φ, ψ induce the bounded operators $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ and $\psi^*\colon L^1_p(G'') \to L^1_p(G')$. The composition $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ is a bounded operator from $L^1_p(G'')$ to $L^1_p(G)$. By Theorem 4.5 the homeomorphism $\psi \circ \varphi$ is a p-quasiconformal homeomorphism. ∎

Example 3: A p-quasiconformal homeomorphism φ with the inverse homeomorphism φ^{-1} which does not have the Hölder property.

Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha = \sum_1^n \alpha_i$ with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Consider the mapping $\varphi\colon B(0,1) \to \mathbb{R}^n$,

$$\varphi(x) = \left(x_1 \cdot e^{-\alpha_1/|x|}, x_2 \cdot e^{-\alpha_2/|x|}, \ldots, x_n \cdot e^{-\alpha_n/|x|}\right).$$

The homeomorphism is p-quasiconformal for $p \geq \alpha/\alpha_1$. The inverse homeomorphism does not have the Hölder property.

Remark: By Corollary 2.7, a p-quasiconformal homeomorphism is locally Lipschitz for $p > n$. By Example 3, for every $p > n$ there exists a p-quasiconformal homeomorphism φ such that the inverse homeomorphism does not have the Hölder property. So the semigroup $\mathrm{QC}_p(G)$ is not a group for $p \neq n$.

PROPOSITION 4.6: *Let $p > n-1$, let G be a domain in $\mathbb{R}^n$ and let $B(x,2R) \subset G$ be a ball. If two continuums F_0, F_1 connect the spheres $S(x,R)$ and $S(x,2R)$, then $\mathrm{Cap}_p(F_0, F_1, G) \geq \alpha^2(n) \cdot R^{n-p}$.*

See, for example, [4], Proposition 6.7 for $n = 2$. The proof for the case $p = n$ is the same. However, the proposition follows from [10], section 9.1.2.

PROPOSITION 4.7: *Let $p > n-1$, let a closed ball $\operatorname{cl} B(x,2R) \subset G$ and $\varphi\colon G \to G'$ be a weakly p-quasiconformal homeomorphism. Then there is a number $Q(n, K_p^1(\varphi))$ such that*

$$\frac{\sup_{z\in S(x,R)} |\varphi(x)-\varphi(z)|^p}{|\varphi(B(x,2R)|} \cdot R^{n-p} \le Q(n, K_p^1(\varphi)).$$

Proof: Let $L = \sup_{y\in S(x,R)} |\varphi(x)-\varphi(y)|$ and $\overline{y}$ be such that $L = |\varphi(x) - \overline{y}|$. We can suppose that $\varphi(x) = (0,0,\dots,0)$ and $\overline{y} = (t_0,\dots,0)$. Suppose that $\overline{y}_1 = (t_1, 0, \dots, 0)$ is such that: (1) $\overline{y_1} \in \varphi(S(x,2r)$, (2) the segment $F_0 = [\overline{y}, \overline{y_1}]$ connects $\varphi(S(x,R))$ and $\varphi(S(x,2R))$. Consider the function $u\colon \mathbb{R} \to \mathbb{R}$ such that: $u(t) = t/L$ for $0 \le t \le t_0$; $u(t) = 0$ for $t < 0$; and $u(t) = 1$ for $t > t_0$. The function $v(y) = u(y_1)$ for $y = (y_1, y_2, \dots, y_n)$ is a Lipschitz function. Let $F_1 = \{\varphi(\operatorname{cl}(B(x,2R) \setminus \varphi(B(X,R)\} \cap \{G' \backslash \operatorname{Int}(\operatorname{Supp} v)\}$. The function v is admissible for the pair F_0, F_1. From the definition of the p-capacity and the inequality $|\nabla v| \le 1/L$, it follows that

$$\begin{aligned}\operatorname{Cap}_p(F_0, F_1, \varphi(B(x,2R)) &\le \|v\|^p_{L_1^p(\varphi(B(x,2R))}\\ &= \int_{\operatorname{Supp}(v)\cap\varphi(B(x,2R))} |\nabla v(y)|^p dy \le \frac{|\varphi(B(x,2R)|}{L^p}.\end{aligned}$$

By the Localization Principle and Proposition 4.1

$$K_p^1(\varphi)\operatorname{Cap}_p(F_0, F_1, \varphi(B(x,2R)) \ge \operatorname{Cap}_p(\varphi^{-1}(F_0), \varphi^{-1}(F_1), B(x,2R)).$$

The continuums $\varphi^{-1}(F_0), \varphi^{-1}(F_1)$ connect the spheres $S(x,R)$, $S(x,,2R)$. By Proposition 4.6

$$\operatorname{Cap}_p\Big(\varphi^{-1}(F_0), \varphi^{-1}(F_1), B(x,2R)\Big) \ge \alpha^2(n) \cdot R^{n-p}.$$

Therefore

$$K_p^1(\varphi)\frac{|\varphi(B(x,2R)|}{L^p} \ge \alpha^2(n) \cdot R^{n-p}. \qquad \blacksquare$$

Remark: The proposition is correct for a ball $B(x,qr)$ if $q > 1$. The proof is the same. Of course, the constant $Q(n, K_p^1(\varphi))$ depends on the choice of the number q.

Proposition 4.7 implies the following result:

THEOREM 4.8: *Let $p > n-1$. A weakly p-quasiconformal homeomorphism $\varphi\colon G \to G'$ is ap-quasiconformal.*

From this theorem and Theorems 3.2 we obtain Theorem 1.3 from Introduction.

THEOREM 4.9: *Let G, G', G'' be domains in R^n and $n-1 < p < n$. If $\varphi\colon G \to G'$ and $\psi\colon G' \to G''$ are ap-quasiconformal homeomorphisms, then the homeomorphism $\psi \circ \varphi$ is an ap-quasiconformal homeomorphism.*

Proof: By Theorem 1.2 the homeomorphisms φ, ψ induce the bounded operators $\varphi^*\colon L^1_p(G') \to L^1_p(G)$ and $\psi^*\colon L^1_p(G'') \to L^1_p(G')$. The composition $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ is a bounded operator from $L^1_p(G'')$ to $L^1_p(G)$. By Theorem 4.11, the homeomorphism $\psi \circ \varphi$ is an *ap*-quasiconformal homeomorphism. ∎

Example 4: A *p*-quasiconformal homeomorphism that does not have the Hölder property ($p < n$).

Consider the mapping $\varphi\colon B(0,1) \to B(0,1)$,

$$\varphi(x) = x \cdot \frac{|x|^{-1}}{\log(e/|x|)}.$$

For a point $x \neq 0$,

$$\det J(x,\varphi) = |x|^{-n} \cdot |\log(|x|)|^{n+1} \quad \text{and} \quad l_\varphi(x) = |x|^{-1} \cdot \log^{-1}(e/|x|).$$

Hence for $1 \le p < n$ and $x \neq 0$,

$$K_p(x,\varphi) = \frac{l_\varphi(x)^p}{\varphi'_v(x)} = |x|^{n-p} \cdot |\log(|x|)|^{p-n+1}.$$

So the homeomorphism $\varphi|B(0,1) \setminus \{0\}$ is a *p*-quasiconformal homeomorphism for all $p < n$. For the zero point

$$\begin{aligned} K_p(x,\varphi) &= \limsup_{r\to 0} \frac{\operatorname{diam}(\varphi(B(0,r)))^p}{|\varphi(B(0,r))|} \cdot r^{n-p} \\ &= \sigma^{-1} \limsup_{r\to 0} |\log(r)|^{n-p} \cdot r^{n-p} = 0. \end{aligned}$$

Example 5: A weakly *p*-quasiconformal homeomorphism that is not *ap*-quasiconformal, $p < n-1$.

Fix a number $q > 1$. Let $\varepsilon > 0$ and $\rho = (\sum_{i=2}^n x_i^2)^{1/2}$. Consider functions $a(t) = qt^{-\varepsilon/(n-1)}$, $s(t) = t$, and

$$k(t) = [a^p(t) \cdot t^{(p-1)(1-\varepsilon)} \cdot s(t)^{1-p}]^{1/(n-p)} \tag{12}$$

where $t \in (0,1)$. Pick a sequence of numbers t_m, $m = 1, 2, \dots$ with the following properties: $0 < t_1 < 1$; $2t_{m+1}^{1-\varepsilon} k(t_{m+1}) = t_m k(t_m)$. Note that $t_m \to 0$ and

$$k(t_m) = q^{p/(n-p)} \cdot t_m^{-\varepsilon(np-n+1)/(n-1)(n-p)} \to \infty$$

when $m \to \infty$.

Divide the ball $B(0, t_1^{1-\varepsilon})$ into the parts $D_{i,m}$, $i = 1, 2, 3, 4$, $m = 1, 2, \dots$ in the following manner:

$$D_m = \{x \mid t_m + a(t_m)\rho \le x_1 \le t_m^{1-\varepsilon} - a(t_m)\rho\},$$

$$D_{1,m} = \left\{x \mid t_m + a(t_m)\rho \le x_1 \le t_m + s(t_m) + \rho \cdot a(t_m)\frac{t_m^{1-\varepsilon} - t_m - 2s(t_m)}{t_m^{1-\varepsilon} - t_m}\right\},$$

$$D_{2,m} = D_m \setminus D_{1,m},\ D_{3,m} = \{x \mid t_m < |x| < t_m^{1-\varepsilon}\} \setminus D_m,$$

$$D_{4,m} = \{x \mid t_{m+1}^{1-\varepsilon} \le |x| \le t_m\}.$$

Define the homeomorphism φ: $B(0, t_1^{1-\varepsilon}) \to R^n$ as follows:

1. $\varphi(0) = 0$;
2. $$\varphi(x) = \left(t_m + a(t_m)\rho + \frac{\frac{1}{2}(x_1 - t_m - a(t_m)\rho)(t_m^{1-\varepsilon} - t_m)}{s(t_m)}, x_2, x_3, \dots, x_n\right) \cdot k(t_m)$$

for $x \in D_{1,m}$;

3. $$\varphi(x) = \left(t_m^{1-\varepsilon} - a(t_m)\rho - \frac{\frac{1}{2}(t_m^{1-\varepsilon} - a(t_m)\rho - x_1)(t_m^{1-\varepsilon} - t_m)}{(t_m^{1-\varepsilon} - t_m - s(t_m))}, x_2, x_3, \dots, x_n\right) \cdot k(t_m)$$

for $x \in D_{2,m}$;

4. $\varphi(x) = k(t_m)x$ for $x \in D_{3,m}$;
5. $$\varphi(x) = \frac{(t_{m+1}^{1-\varepsilon} k(t_{m+1}) + \frac{1}{2}(|x| - t_{m+1}^{1-\varepsilon})t_m \cdot k(t_m)}{(t_m - t_{m+1}^{1-\varepsilon})) \cdot \frac{x}{|x|}}$$

for $x \in D_{4,m}$.

Note that the point $(t_m + s(t_m), 0, \dots, 0) \in D_{1,m}$ maps to the point $(k(t_m)(t_m + t_m^{1-\varepsilon})/2, 0, \dots, 0)$. It is not difficult to see that φ is a homeomorphism.

Let us prove that φ is a weakly p-quasiconformal homeomorphism.

For $x \in D_{1,m}$

$$\det J(x, \varphi) \cong t_m^{1-\varepsilon} \cdot s(t_m)^{-1} \cdot k(t_m)^n$$

and

$$l_\varphi(x) \cong a(t_m) \cdot t_m^{1-\varepsilon} \cdot s(t_m)^{-1} \cdot k(t_m).$$

Hence

$$\frac{l_\varphi(x)^p}{\varphi_v'(x)} \cong a(t_m)^p \cdot t_m^{(1-\varepsilon)(p-1)} \cdot s(t_m)^{1-p} \cdot k(t_m)^{p-n} = 1.$$

For $x \in D_{2,m}$, $\det J(x,\varphi) \cong k(t_m)^n$ and $l_\varphi(x) \cong a(t_m)k(t_m)$. Hence

$$\frac{l_\varphi(x)^p}{\varphi_v'(x)} \cong a(t_m)^p \cdot k(t_m)^{p-n} = t_m^{\varepsilon(p-1)} s^{p-1}(t_m) \to 0$$

when $t_m \to 0$.

In the set $D_{3,m}$ the homeomorphism φ is a homothetic transformation with the coefficient $k(t_m) > 1$. Hence φ is a p-quasiconformal homeomorphism for all $p < n$.

For $x \in D_{4,m}$, $\det J(x,\varphi) \cong k(t_m) \cdot k(t_{m+1})^{n-1}$ and $l_\varphi(x) \cong k(t_{m+1})$. Hence

$$\frac{l_\varphi(x)^p}{\varphi_v'(x)} \cong k(t_m)^{-1} \cdot k(t_{m+1})^{p-n+1} < 1$$

because $p < n-1$.

Let us prove that the homeomorphism φ is not an ap-quasiconformal homeomorphism at $x = 0$. We have

$$l_\varphi(0, t_m + s(t_m)) = \frac{1}{2}(t_m + t_m^{1-\varepsilon})k(t_m)/t_m$$

and

$$\varphi_v(0, q(t_m + s(t_m))) \le \frac{((t_m + s(t_m))^n + (a(t_m)^{-1} q s(t_m))^{n-1} t_m^{1-\varepsilon} \cdot k^n(t_m)}{(t_m + s(t_m))^n}$$
$$\cong k(t_m)^n.$$

So

$$\frac{l_\varphi(0, t_m + s(t_m))^p}{\varphi_v(0, q(t_m + s(t_m))} \cong t_m^{-p\varepsilon} k(t_m)^{p-n} = q^{-p} t_m^{\varepsilon(p-n+1)/(n-1)} \to \infty$$

when $t_m \to 0$.

Example 6: An ap-quasiconformal homeomorphism that is not p-quasiconformal, $p < n$.

Let $a(t) = 1$, $s(t) = t^{1+\varepsilon/(n-1)}$. Define the homeomorphism φ in the same way as in Example 5.

Here we have

$$k(t_m) = t_m^{-n\varepsilon(p-1)/(n-1)(n-p)}. \tag{13}$$

By the calculation of Example 5 the homeomorphism φ is p-quasiconformal in domains $D_{1,m}$, $D_{2,m}$, $D_{3,m}$ for all $p < n$. We must prove that the expression $k(t_m)^{-1} \cdot k(t_{m+1})^{p-n+1}$ is bounded in $D_{4,m}$. From (13) it follows that:

$$k(t_m)^{-1} \cdot k(t_{m+1})^{p-n+1} \leq 2t_{m+1}^{\frac{n\varepsilon(p-1)}{n-1}\left(1-\frac{\varepsilon(n-1)}{(n-p)(n-1)-n\varepsilon(p-1)}\right)}.$$

So for sufficiently small ε the homeomorphism φ is p-quasiconformal in $D_{4,m}$.

At 0,

$$l_\varphi(0, t_m + s(t_m)) \cong t_m^{-\varepsilon} \cdot k(t_m)$$

and

$$\varphi_v(0, t_m + s(t_m)) \cong k(t_m)^n.$$

So

$$\frac{l_\varphi(0, t_m + s(t_m))^p}{\varphi_v(0, t_m + s(t_m))} \cong t_m^{-p\varepsilon} k(t_m)^{p-n} = t_m^{\frac{\varepsilon(p-n)}{n-1}} \to \infty$$

when $t_m \to 0$. So the homeomorphism φ is not p-quasiconformal. At the same time, for $q > 1$, we have $\varphi_v(0, q(t_m + s(t_m)) \cong k^n(t_m) t_m^{-\varepsilon}$. Hence

$$\frac{l_\varphi(0, t_m + s(t_m))^p}{\varphi_v(0, q(t_m + s(t_m))} \cong t_m^{(1-p)\varepsilon} k^{p-n}(t_m) = t_m^{\varepsilon(p-1)/(n-1)} \to 0.$$

4.2 Embedding theorem. The following example will be useful to illustrate the Embedding Theorem (Theorem 4.10).

Example 7: Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha = \sum_1^n \alpha_i$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n = 1$. Consider the domain $K_\alpha = \{x \in \mathbb{R}^n \colon 0 < x_n < 1;\ 0 < x_i < x_n^{\alpha_i}\}$ for $i = 1, 2, \ldots n-1$. Let us denote by K_1 the domain K_α for $\vec{\alpha} = (1, 1, \ldots, 1)$. Let $a > 0$. The homeomorphism $\varphi_a \colon K_1 \to K_\alpha$,

$$\varphi_a(x) = (x_1 \cdot x_n^{a \cdot \alpha_1 - 1},\ x_2 \cdot x_n^{a \cdot \alpha_2 - 1}, \ldots, x_n \cdot x_n^{a \cdot \alpha_n - 1} = x_n^a)$$

has $\det I(x, \varphi_a) = x_n^{a \cdot \alpha - n}$ and $l_{\varphi_a}(x) \leq \text{const} \cdot x_n^{a-1}$. Hence the homeomorphism φ_a is p-quasiconformal for all p such that $p(a-1) \geq a \cdot \alpha - n$. If $a > 1$, then $p \geq (a\alpha - n)/(a-1) \geq \alpha \geq n$. If $a < 1$ and $a \leq (n-1)/(\alpha - 1)$, then $1 \leq p \leq (n - a\alpha)/(1-a)$. So for $p \in [\alpha, \infty) \cup [1, n)$ there is a p-quasiconformal homeomorphism.

Remark: There exists a quasiisometrical homeomorphism from a ball $B(0, r)$ to K_1. So, in Example 7, we can replace the set K_1 by a ball.

THEOREM 4.10: *Let G be a domain in $\mathbb{R}^n$. If there exists a p-quasiconformal homeomorphism φ: $B(0,1) \to G$ for some $p > n$, then $W_p^1(G) \subset C(G)$ and there is a constant Q such that $|\,u|_{C(G)} \leq Q \cdot |\,u|_{W_p^1(G)}$ for all the functions $u \in W_p^1(G)$. The constant Q is independent of the function u.*

Proof: Fix a function $u \in W_p^1(G)$. By Theorem 4.5, $\varphi^*(u) \in L_p^1(B(0,1))$. Hence $\varphi^*(u) \in W_p^1(B(0,1))$. By the classical embedding theorem, $u \circ \varphi \overset{\text{def}}{=} \varphi^*(u) \in C(B(0,1))$. So $u \in C(G)$. The first part of the theorem is proved.

We will use the notation $\operatorname{osc}(u) \overset{\text{def}}{=} \sup_{x\in G} u(x) - \inf_{x\in G} u(x)$. By Theorem 4.5

$$||\varphi^*(u)||_{L_p^1(B(0,1))} \leq ||\varphi^*|| \cdot ||u||_{L_p^1(G))}.$$

Hence

$$||u||_{C(G)} = ||\varphi^*(u)||_{C(B(0,1))} \leq \operatorname{osc}(u) + \inf_{x\in G} u(x).$$

By the Sobolev inequality there is a constant $Q_1(n)$ such that

$$\operatorname{osc}(u) \leq Q_1(n) \cdot ||\varphi^*(u)||_{L_p^1(B(0,1))}.$$

From these three inequalities we obtain

$$\begin{aligned} ||u||_{C(G)} &\leq Q_1(n) \cdot ||\varphi^*|| \cdot ||u||_{L_p^1(G))} + \inf_{x\in G} u(x) \\ &\leq Q_1(n) \cdot ||\varphi^*|| \cdot ||u||_{L_p^1(G))} + |G|^{-1}||u||_{L_p(G)}. \qquad \blacksquare \end{aligned}$$

Remark: By Example 7, for $G = K_\alpha$ the theorem is correct for all $p \geq \alpha$. For $n \leq p < \alpha$ there are unbounded functions in the space $W_p^1(K_\alpha)$. For example, the function $u(x) = x_n^\beta$, $1 - \alpha/p < \beta < 0$ belongs to the space $W_p^1(B(0,1))$. Indeed

$$\begin{aligned} \int_{K_\alpha} |\nabla(u(x))|^p dx &= |\beta|^p \int_{K_\alpha} x_n^{p\cdot(\beta-1)} dx \\ &= |\beta|^p \int_0^1 x_n^{p\cdot(\beta-1)+\alpha-1} dx_n < \infty. \end{aligned}$$

From the theorem and the remark it follows that a p-quasiconformal homeomorphism from a ball $B(x,r)$ to the domain K_α does not exist for $n < p < \alpha$.

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